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Weakly Equilibrium Cantor-type Sets

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Abstract Cantor-type sets are constructed as the intersection of the level domains for simple sequences of polynomials. This allows to obtain Green functions with various moduli of continuity and compact sets with preassigned growth of Markov's factors.

Keywords Green's function · Modulus of continuity · Markov's factors

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1 Introduction

If a compact set $K \subset \mathbb{C}$ is regular with respect to the Dirichlet problem then the Green function $g_{\mathbb{C}\setminus K}$ of $\mathbb{C} \setminus K$ with pole at infinity is continuous throughout $\mathbb{C}$. We are interested in analysis of a character of smoothness of $g_{\mathbb{C}\setminus K}$ near the boundary of K. For example, if $K \subset \mathbb{R}$ then the monotonicity of the Green function with respect to the set K implies that the best possible behavior of $g_{\mathbb{C}\setminus K}$ is $Lip\frac{1}{2}$ smoothness. An important characterization for general compact sets with $g_{\mathbb{C}\setminus K} \in Lip\frac{1}{2}$ was found in [20] by Totik. The monograph [20] revives interest in the problem of boundary behavior of Green functions. Various conditions for optimal smoothness of $g_{\mathbb{C}\setminus K}$ in terms of metric properties of the set K are suggested in [7], and in papers by Andrievskii [2, 3]. On the other hand, compact sets are considered in [1, 8] such that the corresponding Green functions have moduli of continuity equal to some degrees of h, where the function $h(\delta) = (\log \frac{1}{\delta})^{-1}$ defines the logarithmic measure of sets. For a recent result on smoothness of $g_{\mathbb{C}\setminus K_0}$, where K_0 is the classical Cantor ternary set, see [15].

Here the Cantor-type set $K(\gamma)$ is constructed as the intersection of the level domains for a certain sequence of polynomials depending on the parameter

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$\gamma = (\gamma_n)_{n=1}^{\infty}$. In favor of $K(\gamma)$, in comparison to usual Cantor-type sets, it is weakly equilibrium in the following sense.

Consider a Cantor-type set $K = \cap_{s=0}^{\infty} E_s$, where $E_0 = [0, 1]$, E_s is a union of 2^s closed intervals $I_{j,s}$ of positive length, and E_{s+1} is obtained by deleting an open subinterval from each $I_{j,s}$ for $1 \leq j \leq 2^s$. Perhaps the lengths of deleted subintervals are different. Given $s \in \mathbb{N}$, let us uniformly distribute the mass 2^{-s} on each $I_{j,s}$ for $1 \leq j \leq 2^s$. Let us denote by λ_s the normalized in this sense Lebesgue measure on the set E_s. Then, for our case, λ_s converges in the weak* topology to the equilibrium measure $\mu_{K(\gamma)}$ of the set $K(\gamma)$. This is not valid for geometrically symmetric Cantor-type sets (Section 6). If all intervals $(I_{j,s})_{j=1}^{2^s}$ have the same length, that is λ_s is the normalized in the usual sense Lebesgue measure on E_s, then $w^* - \lim \lambda_s$ coincides with the Cantor–Lebesgue measure λ_K. Then the measures μ_K and λ_K are essentially different. Makarov and Volberg proved in [12] for the classical Cantor set K_0 that the carrier of μ_{K_0} has the Hausdorff dimension smaller than $\log 2/ \log 3$. Since λ_{K_0} is just the Hausdorff measure corresponding to this number, the measures λ_{K_0} and μ_{K_0} are mutually singular. For a treatment of a more general case we refer the reader to Chapter IX in [10], see also [4, 21].

Different values of γ provide Green's functions with diverse moduli of continuity (Section 8).

In Section 9 we estimate Markov's factors for the set $K(\gamma)$ and construct a set with preassigned growth of subsequence of Markov's factors.

For basic notions of logarithmic potential theory we refer the reader to [10, 14, 17].

We use the notation $|\cdot|_K$ for the supremum norm on K, log denotes the natural logarithm, $0 \cdot \log 0 := 0$. By $\mathcal{P}_n$ we denote the set of all holomorphic polynomials of degree at most n.

2 Construction of $K(\gamma)$

Suppose we are given a sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < 1/4$. Let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s \in \mathbb{N}$. We define inductively a sequence of real polynomials: let $P_2(x) = x(x-1)$ and $P_{2^{s+1}} = P_{2^s}(P_{2^s} + r_s)$ for $s \in \mathbb{N}$. By that we have a geometric procedure to define new (with respect to P_{2^s}) zeros of $P_{2^{s+1}}$: they are abscissas of points of intersection of the line $y = -r_s$ with the graph $y = P_{2^s}$.

We begin with an elementary lemma which will justify the construction.

Lemma 1 *All critical points of P_{2^s} are decomposed into two groups: 2^{s-1} points of minimum with equal values $P_{2^s} = -r_{s-1}^2/4$ and $2^{s-1} - 1$ points of local maxima with positive values of P_{2^s}. Thus all zeros of P'_{2^s} are simple.*

Proof The proof is by induction on s. If $s = 1$ then the polynomial P_2 has no local maximum, so we begin from $s = 2$ for the basis case. Clearly, the polynomial $P_4(x) = (x^2 - x)(x^2 - x + r_1)$ with $r_1 = \gamma_1 \in (0, 1/4)$ satisfies the statement. Suppose it is valid as well for P_{2^s}. Since $P'_{2^{s+1}} = P'_{2^s}(2\,P_{2^s} + r_s)$, the set of critical points of $P_{2^{s+1}}$ consists of the critical points of P_{2^s} and the solutions of the equation $2\,P_{2^s} + r_s = 0$.

Suppose x is a point of minimum of P_{2^s}. Then $P_{2^s}(x) = -r_{s-1}^2/4$ and, by the choice of the sequence γ, $P_{2^s}(x) + r_s = r_{s-1}^2(\gamma_s - 1/4) < 0$. From this, $P_{2^{s+1}}(x) > 0$. Besides,

$P''_{2^s}(x) > 0$, by the second derivative test. Therefore, $P''_{2^{s+1}}(x) = P''_{2^s}(x)[2\,P_{2^s}(x) + r_s] + 2[P'_{2^s}(x)]^2 = P''_{2^s}(x) r_{s-1}^2(\gamma_s - 1/2) < 0$, so $P_{2^{s+1}}$ has a local maximum at x.

Similarly, if P_{2^s} has a local maximum at x then $P_{2^s}(x) > 0$, by the inductive hypothesis. Here, $P_{2^{s+1}}(x) > 0$ and $P''_{2^{s+1}}(x) < 0$, since $P''_{2^s}(x) < 0$. It follows that the polynomial $P_{2^{s+1}}$ has a local maximum with a positive value at any critical point of P_{2^s}.

It remains to consider the solutions of the equation $2\,P_{2^s} + r_s = 0$. By the inductive hypothesis, the polynomial P_{2^s} has 2^{s-1} points of minimum with equal values $-r_{s-1}^2/4$, whereas at all local maxima P_{2^s} is positive. Therefore the line $y = -r_s/2$ intersects the graph of P_{2^s} at 2^s distinct points. For each such point x we have $P''_{2^{s+1}}(x) = 2[P'_{2^s}(x)]^2 > 0$, so $P_{2^{s+1}}$ has a minimum at x with the value $P_{2^{s+1}}(x) = -r_s/2 \cdot (r_s - r_s/2) = -r_s^2/4$, which is the desired conclusion.

The total number of critical points of $P_{2^{s+1}}$ that we considered above is $2^{s-1} + 2^{s-1} - 1 + 2^s = 2^{s+1} - 1$. Therefore, $P_{2^{s+1}}$ has no other critical points and all zeros of $P'_{2^{s+1}}$ are simple. □

Let E_s denote the set $\{x \in \mathbb{R} : \ P_{2^{s+1}}(x) \le 0\}$. Thus, $E_0 = [0, 1]$ and $E_s = \{x \in \mathbb{R} : -r_s \le \ P_{2^s}(x) \le 0\}$ for $s \in \mathbb{N}$. Lemma 1 and the inequality $r_s < r_{s-1}^2/4$ imply that the set E_s consists of 2^s disjoint closed *basic intervals* $I_{j,s}$. Clearly, $E_{s+1} \subset E_s$. Set $K(\gamma) = \cap_{s=0}^{\infty} E_s$.

3 Location of Zeros

Let $l_{j,s}$ denote the length of the basic interval $I_{j,s}$. In general, the lengths $l_{j,s}$ of intervals of the same level are different, however $\max_{1\le j\le 2^s} l_{j,s} \to 0$ for $s \to \infty$, as we will show in this section.

For fixed $s \in \mathbb{N}$, we enumerate the intervals $(I_{j,s})_{j=1}^{2^s}$ from the left to the right. For example, $I_{1,3} = [0, l_{1,3}]$, $I_{2,3} = [l_{1,2} - l_{2,3}, l_{1,2}]$, $I_{3,3} = [l_{1,1} - l_{2,2}, l_{1,1} - l_{2,2} + l_{3,3}]$, $I_{4,3} = [l_{1,1} - l_{4,3}, l_{1,1}]$, etc.

Let us decompose all zeros of P_{2^s} into s groups. Let $x_1 = 0$, $x_2 = 1$ and $X_0 = \{x_1, x_2\}$. For $k \in \mathbb{N}$ the set $X_k = \{x : \ P_{2^k}(x) + r_k = 0\}$ consists of all zeros of $P_{2^{k+1}}$ that are not zeros of P_{2^k}. Thus, $X_1 = \{x_3, x_4\} = \{l_{1,1}, 1 - l_{2,1}\}, \cdots, X_k = \{x_{2^k+1}, \cdots, x_{2^{k+1}}\} = \{l_{1,k}, l_{1,k-1} - l_{2,k}, \cdots, 1 - l_{2^k,k}\}$. Set $Y_s = \cup_{k=0}^{s} X_k$. Then $P_{2^s}(x) = \prod_{x_k \in Y_{s-1}} (x - x_k)$.

Each $x \in X_s$ has the representation $x = l_{i_1,q_1} - l_{i_2,q_2} + \cdots + (-1)^{m+1} l_{i_m,q_m}$ with $0 \le q_1 < q_2 < \cdots < q_m = s$. The first indices $(i_k)_{k=1}^{m}$ are uniquely defined by the set $(q_k)_{k=1}^{m}$.

Our next goal is to express the values of $x \in X_s$ in terms of the function $u(t) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4t}$ with $0 \le t \le \frac{1}{4}$. Clearly, $u(t)$ and $1 - u(t)$ are the solutions of the equation $P_2(x) + t = 0$. Given $s \in \mathbb{N}$, let us consider the expression

$$x = f_1(\gamma_1 \cdot f_2(\gamma_2 \cdots f_{s-1}(\gamma_{s-1} \cdot f_s(\gamma_s)) \cdots), \tag{1}$$

where $f_k = u$ or $f_k = 1 - u$ for $1 \le k \le s$, so $f_k(t)(1 - f_k(t)) = t$. For each x defined by Eq. 1 we have $P_2(x) = -\gamma_1 \cdot f_2(\gamma_2 \cdots)$ with $\gamma_1 = r_1$. Hence, $P_4(x) = P_2(x)(P_2(x) + r_1) = -r_1^2 f_2(1 - f_2) = -r_1^2 \gamma_2 f_3 = -r_2\, f_3(\gamma_3 \cdots)$. We continue in this fashion to obtain eventually $P_{2^s}(x) = -r_{s-1}^2 \gamma_s = -r_s$, which gives $x \in X_s$.

The formula 1 provides 2^s possible values x. Let us show that they are all different, so any $x_k \in X_s$ can be represented by means of Eq. 1. Since u

increases and $u(a) < 1 - u(b)$ for $a, b \in (0, \frac{1}{4})$, we have $u(\gamma_1 \cdot u(\gamma_2 \cdots \gamma_m u(a)) \cdots) < u(\gamma_1 \cdot u(\gamma_2 \cdots \gamma_m (1 - u(b)) \cdots)$. In general, let $x_i = u(\gamma_1 \cdot u(\gamma_2 \cdots \gamma_{k_1}(1 - u(\gamma_{k_1+1} \cdot u(\cdots \gamma_{k_2}(1 - u(\gamma_{k_2+1} \cdots \gamma_{k_m}(1 - u(a)) \cdots)$ and $x_j = u(\gamma_1 \cdot u(\gamma_2 \cdots \gamma_{k_1}(1 - u(\gamma_{k_1+1} \cdot u(\cdots \gamma_{k_2}(1 - u(\gamma_{k_2+1} \cdots \gamma_{k_m} \cdot u(b)) \cdots)$, that is the first k_m functions f_k for both points are identical, whereas $f_{k_m+1} = 1 - u$ for x_i and u for x_j. The straightforward comparison shows that $x_i > x_j$ for odd m and $x_i < x_j$ otherwise.

There is a simple rule to find, for a given $x \in X_s$, the functions $(f_k)_{k=1}^s$ in Eq. 1. We replace any $l_{i,q}$ with $\gamma_1 \gamma_2 \cdots \gamma_q$. At least for small $(\gamma_k)_{k=1}^q$ this substitution is not rough (Lemma 6 below). Then, for $x = l_{i_1,q_1} - l_{i_2,q_2} + \cdots + (-1)^{m+1} l_{i_m,q_m}$ we have $x \approx \gamma_1 \gamma_2 \cdots \gamma_{q_1}(1 - \gamma_{q_1+1} \cdots \gamma_{q_2}(1 - \gamma_{q_2+1} \cdots \gamma_{q_{m-1}}(1 - \gamma_{q_{m-1}+1} \cdots \gamma_{q_m})) \cdots)$. We put u in front of each γ_k. Thus, in order to get the exact value of x, we have to take all $f_k = u$, except $f_{q_1+1}, \cdots, f_{q_{m-1}+1}$ that are equal to $1 - u$.

For example, $X_3 = \{x_9, \cdots, x_{16}\} = \{l_{1,3}, l_{1,2} - l_{2,3}, l_{1,1} - l_{2,2} + l_{3,3}, \cdots, 1 - l_{8,3}\} = \{u(\gamma_1 \cdot u(\gamma_2 \cdot u(\gamma_3))), u(\gamma_1 \cdot u(\gamma_2 \cdot (1 - u(\gamma_3)))), u(\gamma_1 \cdot (1 - u(\gamma_2 \cdot (1 - u(\gamma_3))))), \cdots\}$.

We use the following properties of the function u:

$$u(t)\sqrt{1-4t} \le t \quad \text{for} \quad 0 \le t \le 1/4, \tag{2}$$

$$u(a\,t) \le a\,u(t) \quad \text{for} \quad 0 \le t \le 1/4,\ 0 \le a \le 1, \tag{3}$$

$$u(b\,t) - u(at) \le 2t\sqrt{b-a} \quad \text{for} \quad 0 \le t \le 1/4,\ 0 \le a < b \le 1 \tag{4}$$

Indeed, the representation $u(t) = \frac{2t}{1+\sqrt{1-4t}}$ implies Eqs. 2 and 3, whereas Eq. 4 is equivalent to $\sqrt{b-a} \le \sqrt{1-4at} + \sqrt{1-4bt}$, which is valid since $0 \le 2(1-b) + (1-4t)(b+a) + 2\sqrt{1-4at}\sqrt{1-4bt}$.

Lemma 2 *Given s, we have* $\min_{1 \le j \le 2^s} l_{j,s} = l_{1,s}$ *and* $\max_{1 \le j \le 2^s} l_{j,s} \le (1/\sqrt{2})^{s+1}$.

Proof Let us show that $l_{1,s} \le l_{j,s}$ for each $s \in \mathbb{N}$ and $1 \le j \le 2^s$. By symmetry, we can suppose that $I_{j,s} = [y, x]$ with $x \in X_s$, $y \in X_m$ where $0 \le m \le s-1$. If $m = 0$ then $I_{j,s} = I_{1,s}$, so we can exclude this case. For $m \ge 1$, from Eq. 1 we have $y = F(\gamma_m)$ with $F(t) = f_1(\gamma_1 \cdot f_2(\gamma_2 \cdots f_{m-1}(\gamma_{m-1} \cdot f_m(t)) \cdots)$ for some $f_k \in \{u, 1-u\}$. Then $x = F(\gamma_m\, a_m)$ with $a_m = 1 - u(\gamma_{m+1} \cdot u(\gamma_{m+2} \cdots u(\gamma_s)) \cdots)$.

By the Mean Value Theorem, $l_{j,s} = x - y = |F'(\xi)| \cdot \gamma_m \cdot u(\gamma_{m+1} \cdots u(\gamma_s)) \cdots)$ with $\gamma_m\, a_m < \xi < \gamma_m$. To simplify notations, we write $t_k = \gamma_k \cdot f_{k+1}(\gamma_{k+1} \cdots \gamma_{m-1} \cdot f_m(\xi)) \cdots)$ and $\tau_k = \gamma_k \cdot u(\gamma_{k+1} \cdots \gamma_{m-1} \cdot u(\xi)) \cdots)$ for $1 \le k \le m-1$. Since $u(\alpha) < 1 - u(\beta)$ for $\alpha, \beta \in (0, \frac{1}{4})$, we have $\tau_k \le t_k$ and $|f_k'(t_k)| = (1-4t_k)^{-1/2} \ge (1 - 4\tau_k)^{-1/2} = u'(\tau_k) \ge u(\tau_k)/\tau_k$, by Eq. 2.

This gives $|F'(\xi)| = |f_1'(t_1)| \cdot \gamma_1 \cdots |f_{m-1}'(t_{m-1})| \cdot \gamma_{m-1} \cdot |f_m'(\xi)| \ge \gamma_1 \cdots \gamma_{m-1} \cdot \frac{u(\tau_1)}{\tau_1} \cdot \frac{u(\tau_2)}{\tau_2} \cdots \frac{u(\tau_{m-1})}{\tau_{m-1}} \cdot \frac{u(\xi)}{\xi}$. Since $\tau_k = \gamma_k \cdot u(\tau_{k+1})$ for $k \le m-2$ and $\tau_{m-1} = \gamma_{m-1} \cdot u(\xi)$, we obtain $|F'(\xi)| \ge \frac{u(\tau_1)}{\xi}$ and

$$l_{j,s} \ge \frac{u(\tau_1)}{\xi} \cdot \gamma_m \cdot u(\gamma_{m+1} \cdots u(\gamma_s)) \cdots) = a \cdot u(\tau_1)$$

with $a = \frac{1}{\xi} \cdot \gamma_m \cdot u(\gamma_{m+1} \cdots u(\gamma_s)) \cdots)$. Applying Eq. 3 yields $l_{1,s} = u(\gamma_1 \cdot u(\gamma_2 \cdots \gamma_{m-1} \cdot u(\xi\, a)) \cdots) \le a\, u(\tau_1)$, that is $l_{1,s} \le l_{j,s}$, which is the desired conclusion.

Our next goal is to estimate $l_{j,s}$ from above. Given s, take $j \le 2^s$ and $I_{j,s} = [y, x]$. Suppose, as above, that $x \in X_s$ and $y \in X_m$ with $0 \le m \le s-1$. Consider first the case $I_{j,s} \subset I_{1,1}$. If $j = 1$ then $l_{1,s} = u(\gamma_1 \cdot u(\gamma_2 \cdots u(\gamma_s)) \cdots) < 2\gamma_1 \cdots 2\gamma_s$, since $u(t) \le 2t$, by Eq. 4 with $a = 0, b = 1$. Therefore, $l_{1,s} < (1/2)^s$.

If $1 < j \le 2^{s-1}$ then for some $f_k \in \{u, 1-u\}$ we have, as above, $y = u(\gamma_1 \cdot f_2(\gamma_2 \cdots f_{m-1}(\gamma_{m-1} \cdot f_m(\gamma_m)) \cdots)$, $x = u(\gamma_1 \cdot f_2(\gamma_2 \cdots f_{m-1}(\gamma_{m-1} \cdot f_m(\gamma_m \cdot a_m)) \cdots)$ with the same a_m as before.

Let us consider two model cases. Let $0 < \gamma \le 1/4$. Suppose first $b = u(\beta) > a = u(\alpha)$. Here, $\gamma\, b \le 1/8$. The derivative $u'(\xi) = (1-4\xi)^{-1/2}$ increases, $u'(1/8) = \sqrt{2}$. Therefore, by the Mean Value Theorem,

$$u(\gamma\, b) - u(\gamma\, a) \le \sqrt{2}\,\gamma\,(b-a).$$

In the second case, let $b = 1 - u(\gamma_p \cdot u(\gamma_{p+1} \cdots u(\gamma_{q-1}\alpha)) \cdots)$, $a = 1 - u(\gamma_p \cdot u(\gamma_{p+1} \cdots u(\gamma_{q-1}\beta)) \cdots)$ with $1/2 \le \alpha = 1 - u(\cdot) < \beta < 1$. Then $1 - 4\gamma\, b \ge 1 - b > \gamma_p \cdot \gamma_{p+1} \cdots \gamma_{q-1}/2$ as $u(t) > t$. Here, $u(\gamma\, b) - u(\gamma\, a) < u'(\gamma\, b)\,\gamma\,(b-a) \le$

$$\gamma\,\sqrt{2}\,\frac{u(\gamma_p \cdot u(\gamma_{p+1} \cdots u(\gamma_{q-1}\beta)) \cdots) - u(\gamma_p \cdot u(\gamma_{p+1} \cdots u(\gamma_{q-1}\alpha)) \cdots)}{\sqrt{\gamma_p \cdot \gamma_{p+1} \cdots \gamma_{q-1}}}.$$

Arguing as in the first case, we see that the numerator does not exceed the value $(\sqrt{2}\,\gamma_p) \cdots (\sqrt{2}\,\gamma_{q-2})\,[u(\gamma_{q-1}\beta) - u(\gamma_{q-1}\alpha)]$. Therefore,

$$u(\gamma\, b) - u(\gamma\, a) < \sqrt{2} \cdot \gamma \cdot \sqrt{2\,\gamma_p} \cdots \sqrt{2\,\gamma_{q-2}}\;\frac{u(\gamma_{q-1}\beta) - u(\gamma_{q-1}\alpha)}{\sqrt{\gamma_{q-1}}}.$$

We proceed to estimate $x - y$. Since $y \ne 0$, at least one function f_k in the representation of y is $1-u$. Let $f_p = f_q = f_r = \cdots = f_n = 1-u$ for some indexes $2 \le p < q < r < \cdots n \le m$, whereas all other functions in the representation of y are equal to u. Thereby, $x - y = u(\gamma_1 \cdot u(\gamma_2 \cdots u(\gamma_{p-1}(1 - u(\gamma_p \cdots u(\gamma_m a_m)) \cdots) - u(\gamma_1 \cdot u(\gamma_2 \cdots u(\gamma_{p-1}(1 - u(\gamma_p \cdots u(\gamma_m)) \cdots)$. As in the first model case, $x - y \le$

$$\sqrt{2}\gamma_1 \cdots \sqrt{2}\gamma_{p-2}\,[u(\gamma_{p-1}(1 - u(\gamma_p \cdots u(\gamma_m a_m)) \cdots) - u(\gamma_{p-1}(1 - u(\gamma_p \cdots u(\gamma_m)) \cdots)].$$

We apply the second model case with $b = 1 - u(\gamma_p \cdot u(\gamma_{p+1} \cdots u(\gamma_{q-1} a_{q-1})) \cdots)$ where $a_{q-1} = 1 - u(\gamma_q \cdots u(\gamma_m a_m)) \cdots)$ and $a = 1 - u(\gamma_p \cdots u(\gamma_{q-1} b_{q-1})) \cdots)$, $b_{q-1} = 1 - u(\gamma_q \cdots u(\gamma_m)) \cdots)$. This gives

$$x - y \le \sqrt{2}\gamma_1 \cdots \sqrt{2}\gamma_{p-2}\,\sqrt{2}\gamma_{p-1} \cdot \sqrt{2\,\gamma_p} \cdots \sqrt{2\,\gamma_{q-2}}\,\frac{u(\gamma_{q-1} b_{q-1}) - u(\gamma_{q-1} a_{q-1})}{\sqrt{\gamma_{q-1}}}.$$

Repeating this argument for the numerator leads to

$$u(\gamma_{q-1} b_{q-1}) - u(\gamma_{q-1} a_{q-1}) \le \sqrt{2}\gamma_{q-1}\sqrt{2\,\gamma_q} \cdots \sqrt{2\,\gamma_{r-2}}\,\frac{u(\gamma_{r-1} b_{r-1}) - u(\gamma_{r-1} a_{r-1})}{\sqrt{\gamma_{r-1}}}$$

with the corresponding values of b_{r-1} and a_{r-1}. Therefore,

$$x - y \le \left(\sqrt{2}\gamma_1 \cdots \sqrt{2}\gamma_{p-1}\right) \cdot \left(\sqrt{2\,\gamma_p} \cdots \sqrt{2\,\gamma_{r-2}}\right)\frac{u(\gamma_{r-1} b_{r-1}) - u(\gamma_{r-1} a_{r-1})}{\sqrt{\gamma_{r-1}}}.$$

We continue in this fashion to obtain eventually,

$$\frac{u(\gamma_{n-1}(1-u(\gamma_n\cdots u(\gamma_m a_m))\cdots)-u(\gamma_{n-1}(1-u(\gamma_n\cdots u(\gamma_m))\cdots)}{\sqrt{\gamma_{n-1}}}$$
$$\le \sqrt{2\,\gamma_{n-1}}\cdots\sqrt{2\,\gamma_{m-1}}\;\frac{u(\gamma_m)-u(\gamma_m a_m)}{\sqrt{\gamma_m}}.$$

For the last numerator we use Eq. 4: $u(\gamma_m)-u(\gamma_m a_m)\le 2\,\gamma_m\sqrt{1-a_m}$, where $1-a_m=u(\gamma_{m+1}\cdot u(\gamma_{m+1}\cdots u(\gamma_s))\cdots)\le(1/2)^{s-m}$, since $u(t)\le 2t$.

Combining these inequalities gives

$$x-y\le(\sqrt{2}/4)^{p-1}\,(1/\sqrt{2})^{m-p}\,2\sqrt{\gamma_m}\,(1/\sqrt{2})^{s-m}\le(1/\sqrt{2})^{s+2p-3}\le(1/\sqrt{2})^{s+1},$$

since $p\ge 2$.

Similar arguments apply to the case $I_{j,s}\subset I_{2,1}$ with $f_1=1-u$. □

4 The Green Function

Here we consider P_{2^s} as a polynomial of a complex variable.

Lemma 3 *Given* $z\in\mathbb{C}$ *and* $s\in\mathbb{N}$, *let* $w_s=2\,r_s^{-1}\,P_{2^s}(z)+1$. *Suppose* $|w_s|=1+\varepsilon$ *for some* $\varepsilon>0$. *Then* $|w_{s+1}|>1+4\,\varepsilon$.

Proof We have $P_{2^s}=(w_s-1)\,r_s/2$, $P_{2^{s+1}}=P_{2^s}(P_{2^s}+r_s)=(w_s^2-1)\,r_s^2/4$ and $w_{s+1}=(2\,\gamma_{s+1})^{-1}\,(w_s^2-1+2\,\gamma_{s+1})$. Therefore, $|w_{s+1}|$ attains its minimal value on the set $\{w_s:|w_s|=1+\varepsilon\}$ at the point $w_s=1+\varepsilon$, so $|w_{s+1}|\ge(2\,\gamma_{s+1})^{-1}\,(2\,\varepsilon+\varepsilon^2+2\,\gamma_{s+1})>1+\varepsilon/\gamma_{s+1}>1+4\,\varepsilon$. □

Let $D_s=\{z\in\mathbb{C}:|P_{2^s}(z)+r_s/2|<r_s/2\}$. Recall that $K(\gamma)=\cap_{s=0}^{\infty}E_s$ with $E_s=\overline{D}_s\cap\mathbb{R}$. Let us show that $(\overline{D}_s)_{s=1}^{\infty}$ is a nested family.

Theorem 1 *We have* $\overline{D}_s\searrow K(\gamma)$.

Proof The embedding $\overline{D}_{s+1}\subset\overline{D}_s$ is equivalent to the implication

$$|P_{2^s}(z)+r_s/2|>r_s/2\implies|P_{2^{s+1}}(z)+r_{s+1}/2|>r_{s+1}/2,$$

which we have by Lemma 3.

For each $j\le 2^s$ the real polynomial P_{2^s} is monotone on $I_{j,s}$ and takes values 0 and $-r_s$ at its endpoints. Therefore, $E_s\subset\overline{D}_s$ and $K(\gamma)\subset\cap_{s=0}^{\infty}\overline{D}_s$.

For the inverse embedding, let us fix $z\notin K(\gamma)$. We need to find s with $z\notin\overline{D}_s$. Assume first $z\in\mathbb{R}$. Since $\overline{D}_s\cap\mathbb{R}=E_s$, the condition $z\notin E_s$ gives the desired s.

Let $z=x+i\,y$ with $y\ne 0, x\notin K(\gamma)$. By the above, $x\notin\overline{D}_s$ for some s. All zeros $(c_j)_{j=1}^{2^s}$ of the polynomial $P_{2^s}+r_s/2$ are real. Therefore, $|z-c_j|>|x-c_j|$ and $|P_{2^s}(z)+r_s/2|>|P_{2^s}(x)+r_s/2|>r_s/2$, so $z\notin\overline{D}_s$.

It remains to consider the case $z=x+i\,y$ with $y\ne 0, x\in K(\gamma)$. If $|y|\ge 2$ then $z\notin\overline{D}_1$. Indeed, $|P_2(z)+r_1/2|\ge|Re(P_2(z)+r_1/2)|=y^2-x^2+x-r_1/2$, which exceeds $r_1/2$, since $0\le x\le 1$ and $r_1=\gamma_1<1/4$.

Let $0 < |y| < 2$. By Lemma 2, we can choose s such that $\max_{1\le j\le 2^s} l_{j,s} < y^2/2$. Given s, fix k with $x \in I_{k,s} = [a, b]$. Here, $|P_{2^s}(a) + r_s/2| = r_s/2$. Let us show that $|P_{2^s}(z) + r_s/2| > |P_{2^s}(a) + r_s/2|$ by comparison the distances from z and from a to the point c_j.

If $j < k$ then $|a - c_j| \le |x - c_j|$, which is less than the hypotenuse $|z - c_j|$.

If $j = k$ then $|a - c_k| < l_{k,s} < y^2/2 < |y| \le |z - c_k|$, by the choice of s.

If $j > k$ then $c_j - a = c_j - b + l_{k,s}$. Therefore, $|c_j - a|^2 = |c_j - b|^2 + 2l_{k,s}(c_j - b + l_{k,s}/2) < |c_j - b|^2 + 2l_{k,s}$, since $c_j - b + l_{k,s}/2 < c_j - a < 1$. As above, $2l_{k,s} < y^2$. It follows that $|c_j - a|^2 < |c_j - b|^2 + y^2 \le |c_j - x|^2 + y^2 = |z - c_j|^2$. □

Corollary 1 *The set $K(\gamma)$ is polar if and only if $R := \lim_{s\to\infty} 2^{-s} \log \frac{2}{r_s} = \infty$. If this limit is finite and $z \notin K(\gamma)$, then*

$$g_{\mathbb{C}\setminus K(\gamma)}(z) = \lim_{s\to\infty} 2^{-s} \log |P_{2^s}(z)/r_s|.$$

Proof Suppose $P \in \mathcal{P}_n$ has a leading coefficient a_n and $\Omega = \{z : |P(z)| > 1\}$. Then, clearly, $g_\Omega(z) = n^{-1} \log |P(z)|$ with the corresponding Robin constant equals to $n^{-1} \log |a_n|$. In our case, $g_{\mathbb{C}\setminus\overline{D}_s}(z) = 2^{-s} \log |2r_s^{-1} P_{2^s}(z) + 1|$ and $R_s := Rob(\overline{D}_s) = 2^{-s} \log \frac{2}{r_s}$. Since the sequence $(R_s)_{s=1}^\infty$ increases to R, the infinite value of R gives polarity of $K(\gamma)$.

If R is finite then, by the Harnack Principle (see e.g. [17], Theorem 0.4.10), $g_{\mathbb{C}\setminus\overline{D}_s} \nearrow g_{\mathbb{C}\setminus K(\gamma)}$ uniformly on compact subsets of $\mathbb{C} \setminus K(\gamma)$. Suppose $z \notin K(\gamma)$. Then $z \notin \overline{D}_q$ for some $q \in \mathbb{N}$. Fix $\varepsilon > 0$ with $|2r_q^{-1} P_{2^q}(z) + 1| = 1 + \varepsilon$. By Lemma 3, $|2r_s^{-1} P_{2^s}(z) + 1| > 1 + 4^{s-q}\varepsilon$, so, for large s, the value $|P_{2^s}(z)/r_s|$ dominates 1. This gives the desired representation of $g_{\mathbb{C}\setminus K(\gamma)}$. □

Recall that a monic polynomial $P \in \mathcal{P}_n$ is a Chebyshev polynomial for a compact set K if the value $|P|_K$ is minimal among all monic polynomials of degree n.

The next proposition is a consequence of the Kolmogorov criterion ([11], see also [9], Theorem 3.2.1). We formulate its polynomial version for the case when K is a compact subset of $\mathbb{C}$:

Theorem 2 (Kolmogorov) *A polynomial $P \in \mathcal{P}_n$ is a best approximation to $f \in C(K)$ if and only if for each $Q \in \mathcal{P}_n$ we have $\max_{z\in K_0} Re\{[f(z) - P(z)]\,\overline{Q(z)}\} \ge 0$, where $K_0 = \{z \in K : |f(z) - P(z)| = |f - P|_K\}$.*

Proposition 1 *The polynomial $P_{2^s} + r_s/2$ is the Chebyshev polynomial for $K(\gamma)$.* □

Proof In our case, $f = z^{2^s}$ and $n = 2^s - 1$. We want to show that the polynomial $P = f - P_{2^s} - r_s/2$ is a best approximation to f out of $\mathcal{P}_n$. By Theorem 2, it suffices to show that $\max_{z\in K_0} Re\{[P_{2^s}(z) + r_s/2]\,\overline{Q(z)}\} \ge 0$ for each $Q \in \mathcal{P}_n$. Here, K_0 consists of endpoints of the intervals $I_{j,s}$ for $1 \le j \le 2^s$. Fix $Q \in \mathcal{P}_n$. Then $Q(x + iy) = u(x, y) + i\,v(x, y)$ and $P_{2^s} + r_s/2 = A + i\,B$ for certain real polynomials u, v, A, B of degree n. All coefficients of P_{2^s} are real, so $B(z) = 0$ for real z. In particular, $B = 0$ on K_0.

In these notations, $Re\{[P_{2^s} + r_s/2]\,\overline{Q}\} = A\,u + B\,v = A\,u$ on K_0. Suppose, contrary to our claim, that $A\,u < 0$ on K_0. Since $A(x, 0)$ takes values $\pm r_s/2$ of different signs at endpoints of each interval $I_{j,s}$, the real polynomial $u(\cdot, 0)$ has at least one zero on $I_{j,s}$ for $1 \le j \le 2^s$. But the degree of $u(\cdot, 0)$ does not exceed $2^s - 1$. Therefore, $u(\cdot, 0) \equiv 0$ and $A\, u\,|_{K_0} = 0$, a contradiction. □

Example 1 The statement above is valid as well in the limit case, when $\gamma_s = 1/4$ for all s. Here, $r_s = r_{s-1}^2/4$. By arguments of Lemma 1, $P_{2^s} \le 0$ on $[0, 1]$ for all s, so $K(\gamma) = [0, 1]$. Let T_n be the classical Chebyshev polynomial, that is $T_n(t) = \cos(n \arccos t)$ for $|t| \le 1$. The leading coefficient of T_n for $n \ge 1$ is 2^{n-1}. Therefore, $2^{1-n} T_n$ and $Q_n(z) = 2^{1-2n}\, T_n(2z - 1)$ are the the $n-$th Chebyshev polynomials for $[-1, 1]$ and, respectively, for $[0, 1]$. In particular, $T_2(t) = 2t^2 - 1$. Therefore, $Q_2(z) = z^2 - z + 1/8 = P_2(z) + r_1/2$. By induction, using the relation $T_{2^{s+1}} = T_2(T_{2^s})$, one can easily show that $P_{2^s}(z) + r_s/2 = 2^{1-2^{s+1}}\, T_{2^s}(2z - 1)$ for all $s \in \mathbb{N}$.

5 Auxiliary Results

Recall that $X_0 = \{0, 1\}$, $X_k = \{x : \ P_{2^k}(x) = -r_k\}$ for $k \ge 1$, and $Y_s = \cup_{k=0}^{s} X_k$ is the set of zeros for $P_{2^{s+1}}$.

Since $P'_{2^s} = P'_{2^{s-1}}(2\, P_{2^{s-1}} + r_{s-1})$ for $s \ge 2$, we have

$$P'_{2^s}(y) = r_{s-1}\, P'_{2^{s-1}}(y), \ y \in Y_{s-2}; \ \ P'_{2^s}(x) = -r_{s-1} P'_{2^{s-1}}(x), \ x \in X_{s-1}. \tag{5}$$

After iteration this gives

$$|P'_{2^s}(x)| = r_{s-1}\, r_{s-2} \cdots r_q\, |P'_{2^q}(x)| \quad \text{for} \quad x \in X_q \quad \text{with} \quad q < s. \tag{6}$$

From here, for example, $|P'_{2^s}(0)| = r_{s-1}\, r_{s-2} \cdots r_1$.

The identity $P_{2^{s+1}}(y) = P_{2^s}(y) \prod_{x_k \in X_s}(y - x_k) = P_{2^s}(y)\,(P_{2^s}(y) + r_s)$ implies $P_{2^s}(y) + r_s = \prod_{x_k \in X_s}(y - x_k)$. Thus,

$$\prod_{x_k \in X_s} (y - x_k) = r_s \quad \text{for} \quad y \in Y_{s-1}. \tag{7}$$

From now on we make the assumption

$$\gamma_s \le 1/32 \qquad \text{for} \qquad s \in \mathbb{N}. \tag{8}$$

Each $I_{j,s}$ contains two *adjacent* basic subintervals $I_{2j-1,s+1}$ and $I_{2j,s+1}$. Let $h_{j,s} = l_{j,s} - l_{2j-1,s+1} - l_{2j,s+1}$ be the distance between them.

Lemma 4 *Suppose γ satisfies Eq. 8. Then the polynomial P_{2^s} is convex on $I_{j,s-1}$. For $1 \le j \le 2^{s-1}$ we have $l_{2j-1,s} + l_{2j,s} < 4\,\gamma_s\, l_{j,s-1}$. Thus, $h_{j,s-1} > (1 - 4\gamma_s)\, l_{j,s-1}$.*

Proof We proceed by induction. If $s = 1$ then P_2 is convex on $I_{1,0} = [0, 1]$. Let us show that $l_{1,1} + l_{2,1} < 4\,\gamma_1$. The triangle Δ with the vertices $(0, 0), (1, 0), (\frac{1}{2}, -\frac{1}{4})$ is entirely situated in the epigraph $\{(x, y) \in \mathbb{R}^2 : \ P_2(x) \le y\}$. The line $y = -r_1$ intersects Δ along the segment $[A, B]$. By convexity of P_2, we have $h_{1,0} = 1 - l_{1,1} - l_{2,1} > |B - A|$. The triangle Δ_1 with the vertices $A, B, (\frac{1}{2}, -\frac{1}{4})$ is similar to Δ. Therefore, $\frac{1}{4}\ |B - A| = \frac{1}{4} - r_1$. Here, $r_1 = \gamma_1$, and the result follows.

Suppose we have convexity of $P_{2^k}|_{I_{j,k-1}}$ and the desired inequalities for $k \le s - 1$. Fix $j \le 2^{s-1}$ and $x \in I_{j,s-1} = [a, b]$. Then $P_{2^s}(x) = (x - a)(x - b)\, g(x)$, where $g(x) = \prod_{k=1}^{n}(x - z_k)$ with $n = 2^s - 2$ and $(z_k)_{k=1}^{n} = Y_{s-1} \setminus \{a, b\}$.

Hence,

$$P''_{2^s}(x) = g(x)\left[2 + 2\sum_{k=1}^{n}\frac{2x-a-b}{x-z_k} + \sum_{k=1}^{n}\sum_{i=1,i\neq k}^{n}\frac{(x-a)(x-b)}{(x-z_k)(x-z_i)}\right].$$

Clearly, $g|_{I_{j,s-1}} > 0$, $|2x-a-b| \le l_{j,s-1}$, and $|(x-a)(x-b)| \le \frac{1}{4}\, l_{j,s-1}^2$. For convexity of $P_{2^s}|_{I_{j,s-1}}$ we only need to check that $8 \ge 8\, l_{j,s-1}\sum_{k=1}^{n}|x-z_k|^{-1} + l_{j,s-1}^2\sum_{k=1}^{n}\sum_{i\neq k}|x-z_k|^{-1}|x-z_i|^{-1}$.

Let us consider the basic intervals containing $x:\ I_{j,s-1}\subset I_{m,s-2}\subset I_{q,s-3}\subset\cdots\subset I_{1,0}$. The interval $I_{m,s-2}$ contains two zeros of g. For each of them $|x-z_k| \ge h_{m,s-2} > (1-4\gamma_{s-1})\, l_{m,s-2}$ and $\frac{l_{j,s-1}}{|x-z_k|} < \frac{4\gamma_{s-1}}{1-4\gamma_{s-1}}$, by inductive hypothesis. The last fraction does not exceed 1/7. Similarly, $I_{q,s-3}$ contains another four zeros of g with $\frac{l_{j,s-1}}{|x-z_k|} < \frac{4\gamma_{s-1}\, 4\gamma_{s-2}}{1-4\gamma_{s-2}} \le \frac{1}{7}\cdot\frac{1}{8}$. We continue in this fashion to obtain $l_{j,s-1}\sum_{k=1}^{n}|x-z_k|^{-1} < \sum_{k=1}^{s-1}2^k\cdot\frac{1}{7}\cdot(\frac{1}{8})^{k-1} < \frac{8}{21}$.

In the same way, $l_{j,s-1}^2\sum_{k=1}^{n}\sum_{i\neq k}|x-z_k|^{-1}|x-z_i|^{-1} < (\frac{8}{21})^2$, which gives $P''_{2^s}|_{I_{j,s-1}} > 0$. Arguing as above, by means of convexity of $P_{2^s}|_{I_{j,s-1}}$, it is easy to show the second statement of Lemma. □

Let $\delta_s = \gamma_1\gamma_2\cdots\gamma_s$, so $r_1 r_2\cdots r_{s-1}\,\delta_s = r_s$.

Lemma 5 *Suppose γ satisfies Eq. 8 and I is the basic interval of the $s-$th level with the endpoints $y\in Y_{s-1}$, $x\in X_s$. Then*

$$\exp(-16\,\gamma_s)\;|P'_{2^s}(y)| < |P'_{2^s}(x)| < |P'_{2^s}(y)| = \max_{t\in I}|P'_{2^s}(t)|.$$

Proof The interval I is a subset of some $I_{j,s-1} = [a,b]$, where, by Lemma 4, the polynomial P_{2^s} is convex, so P'_{2^s} increases. In addition, $P_{2^s}(a) = P_{2^s}(b) = 0$ and, by Lemma 1, P'_{2^s} has one zero ξ with $\min_{t\in I_{j,s-1}}P_{2^s}(t) = P_{2^s}(\xi) = -r_{s-1}^2/4$. The value $P_{2^s}(x)$, that is $-r_s$, is greater than $P_{2^s}(\xi)$. Therefore, $\xi\notin I$ and $|P'_{2^s}|$ attains its maximal value on I at the endpoint from Y_{s-1}. Thus, $|P'_{2^s}(x)| < |P'_{2^s}(y)|$.

In order to get the corresponding lower bound, let us assume, without loss of generality, that $I_{i,s} = [y,x]$ with $y\in Y_{s-1}$, $x = y + l_{i,s}\in X_s$. The point x is a zero of $P_{2^{s+1}}$ and $P'_{2^{s+1}}(x) > 0$. Therefore,

$$P'_{2^{s+1}}(x) = (x-y)\cdot\prod_{y_k\in Y'_s}|x-y_k| = (x-y)\cdot\prod_{y_k\in Y'_s}|y-y_k|\cdot\beta,$$

where $Y'_s = Y_s\setminus\{x,y\}$, $\beta = \prod_{y_k\in Y'_s}\left(1+\frac{l_{i,s}}{y-y_k}\right)$. Here,

$$(x-y)\cdot\prod_{y_k\in Y'_s}|y-y_k| = \prod_{x_k\in X_s}|y-x_k|\prod_{y_k\in Y_{s-1},\,y_k\neq y}|y-y_k| = r_s\,|P'_{2^s}(y)|,$$

by Eq. 7. On the other hand, by Eq. 5, $P'_{2^{s+1}}(x) = r_s\,|P'_{2^s}(x)|$, so $|P'_{2^s}(x)| = \beta\,|P'_{2^s}(y)|$. Let us estimate β from below. We can take into account only $y_k\in Y'_s$ with $y_k > y$, since otherwise the corresponding term in β exceeds 1 and we can neglect it. The interval $I_{j,s-1}$ contains two points y_k with $y_k - y > h_{j,s-1}$. Lemma 4 yields $1+\frac{l_{i,s}}{y-y_k} > 1-\frac{8}{7}\cdot\frac{l_{i,s}}{l_{i,s-1}} > 1-\frac{8}{7}\cdot 4\gamma_s$.

For the next four points (let $I_{j,s-1} \subset I_{m,s-2}$) we have $y_k - y > h_{m,s-2}$ and $1 + \frac{l_{i,s}}{y-y_k} > 1 - \frac{8}{7} \cdot \frac{l_{i,s}}{l_{m,s-2}} > 1 - \frac{8}{7} \cdot 4\gamma_s \cdot 4\gamma_{s-1} \geq 1 - \frac{1}{7} \cdot 4\gamma_s$, by Eq. 8.

We continue in this fashion obtaining $\log\beta > \sum_{k=1}^{s} 2^k \log(1 - \frac{4}{7} \cdot 8^{2-k}\gamma_s)$. If $0 < a < \frac{1}{4}$ then $\log(1-a) > 4\,a\,\log\frac{3}{4} > -1.16\,a$. A straightforward calculation shows that $\log\beta > -16\,\gamma_s$. This gives the desired result. □

Lemma 6 *Let* γ *satisfy Eq.* 8 *and* $x \in X_s$*. Then*

$$\exp\left(-16\sum_{k=1}^{s}\gamma_k\right) \cdot r_s/\delta_s < |P'_{2^s}(x)| \leq |P'_{2^s}|_{E_s} = r_s/\delta_s$$

and

$$\delta_s < l_{i,s} < \exp\left(16\sum_{k=1}^{s}\gamma_k\right) \cdot \delta_s \quad \textit{for} \quad 1 \leq i \leq 2^s.$$

Proof Fix $x \in X_s$. By symmetry, let $x \in I_{1,1}$. Suppose, as in the previous lemma, that x is the right endpoint of some $I_{i,s}$. Then $x = l_{i_1,p} - l_{i_2,q} + \cdots + l_{i_{k-2},m} - l_{i_{k-1},n} + l_{i_k,s}$ with $1 \leq p < q < \cdots m < n < s$. Clear, $i_1 = 1$. By Lemma 5 and Eq. 6, we conclude that $|P'_{2^s}(x)| < |P'_{2^s}(y)| = r_{s-1} \cdot r_{s-2} \cdots r_n \cdot |P'_{2^n}(y)|$.

We apply again Lemma 5 for with y instead of x and $z = l_{i_1,p} - l_{i_2,q} + \cdots + l_{i_{k-2},m} \in X_m$ instead of y to obtain $|P'_{2^n}(y)| < |P'_{2^n}(z)|$. By Eq. 6, $|P'_{2^n}(z)| = r_{n-1} \cdot r_{n-2} \cdots r_m \cdot |P'_{2^m}(z)|$. Similar arguments apply for z, et cetera. Finally, $|P'_{2^s}(x)| < r_{s-1} \cdot r_{s-2} \cdots r_1 = r_s/\delta_s$ if $p > 1$ or $|P'_{2^s}(x)| < r_s/\delta_s \cdot |P'_2(l_{1,1})|$ if $p = 1$. In the last case, $|P'_2(l_{1,1})| = 1 - 2l_{1,1} < 1$. This gives the desired upper bound.

The lower bound of $|P'_{2^s}(x)|$ can be obtained in the same manner as above, by repeated application of Lemma 5 and Eq. 6. In the worst case, when $p = 1$, $q = 2, \cdots, m = s-2$, $n = s-1$, we have $|P'_{2^s}(x)| > e^{-16\gamma_s} \cdot r_{s-1} \cdot |P'_{2^{s-1}}(y)| > \cdots > e^{-16(\gamma_s + \cdots \gamma_2)} r_{s-1} \cdots r_1 \cdot |P'_2(l_{1,1})|$. Since $|P'_2(l_{1,1})| = \sqrt{1 - 4\gamma_1} > e^{-16\gamma_1}$, the result follows.

The second statement of Lemma can be obtained by the Mean Value Theorem, since $P_{2^s}(y) = 0$, $P_{2^s}(y + l_{i,s}) = -r_s$. In particular, if $x = l_{1,s}$ and $y = 0$ then $\exp(-16\,\gamma_s) \cdot r_s/\delta_s < |P'_{2^s}(x)|$. Therefore,

$$\delta_s < l_{1,s} < \delta_s \cdot e^{16\,\gamma_s} < 2\,\delta_s. \square \tag{9}$$

Corollary 2 *If* γ *satisfies Eq.* 8 *and* $I_{i,s} \subset I_{j,s-1}$ *then* $\frac{1}{2}\,\gamma_s\,l_{j,s-1} < l_{i,s} < 4\,\gamma_s\,l_{j,s-1}$.

Proof The right inequality is given by Lemma 4. To deal with the left one, let us denote by x, y the endpoints of $I_{i,s}$ with $x \in X_s$, $y \in Y_{s-1}$.

Suppose first that $y \in X_{s-1}$. By the Mean Value Theorem, $l_{i,s}\,|P'_{2^s}(\xi)| = r_s$ for some $\xi \in I_{i,s}$. By Lemma 5, $|P'_{2^s}(\xi)| < |P'_{2^s}(y)|$, which is $r_{s-1}\,|P'_{2^{s-1}}(y)|$, by Eq. 5. Here, $|P'_{2^{s-1}}(y)| < |P'_{2^{s-1}}(z)|$, where $z \in Y_{s-2}$ is another endpoint of $I_{j,s-1}$. Therefore, $l_{i,s} > \gamma_s\,r_{s-1}/|P'_{2^{s-1}}(z)|$. On the other hand, $l_{j,s-1} = r_{s-1}/|P'_{2^{s-1}}(\eta)|$ with $\eta \in I_{j,s-1}$, so $|P'_{2^{s-1}}(\eta)| > |P'_{2^{s-1}}(z)|\,e^{-16\,\gamma_{s-1}}$, by Lemma 5. For this reason, $l_{i,s} > \gamma_s\,l_{j,s-1} e^{-16\,\gamma_{s-1}} \geq \frac{1}{2}\,\gamma_s\,l_{j,s-1}$.

The case $y \in Y_{s-2}$ is very similar. Here at once y plays the role of z. □

Beardon and Pommerenke introduced in [5] the concept of uniformly perfect sets. A dozen of equivalent descriptions of such sets are suggested in [10, p. 343]. We use the following: a compact set $K \subset \mathbb{C}$ is *uniformly perfect* if K has at least two points and there exists $\varepsilon_0 > 0$ such that for any $z_0 \in K$ and $0 < r \leq diam(K)$ the set $K \cap \{z : \varepsilon_0 r < |z - z_0| < r\}$ is not empty.

Theorem 3 *The set $K(\gamma)$, provided Eq. 8, is uniformly perfect if and only if* $\inf \gamma_s > 0$.

Proof Suppose $K(\gamma)$ is uniformly perfect. The values $z_0 = 0$ and $r = l_{1,s-1} - l_{2,s}$ in the definition above imply $l_{1,s} + l_{2,s} > \varepsilon_0 \, l_{1,s-1}$. By Lemma 4, we have $4\gamma_s > \varepsilon_0$, so $\inf_s \gamma_s \geq \varepsilon_0/4$, which is our claim.

The converse follows immediately by Corollary 2. □

6 $K(\gamma)$ is Weakly Equilibrium

Here and in the sequel we consider r_s in the form $r_s = 2\exp(-R_s \cdot 2^s)$. Recall that, for $s \in \mathbb{N}$, the value R_s gives the Robin constant for $\overline{D}_s$ and $R_s \uparrow R$, which is finite if $K(\gamma)$ is not a polar set. In this case, let $\rho_s = R - R_s$. Since $r_0 = 1$, we take $\rho_0 = R - \log 2$. In term of $(\gamma_k)_{k=1}^{\infty}$ we have $r_s = \gamma_s \gamma_{s-1}^2 \cdots \gamma_1^{2^{s-1}}$, so $R = \sum_{k=1}^{\infty} 2^{-k} \log \frac{1}{\gamma_k}$ and $\rho_s = \sum_{k=s+1}^{\infty} 2^{-k} \log \frac{1}{2\gamma_k}$. On the other hand, in terms of $(\rho_k)_{k=0}^{\infty}$, we obtain that $\gamma_s = r_s r_{s-1}^{-2} = \frac{1}{2}\exp[2^s(R_{s-1} - R_s)] = \frac{1}{2}\exp[2^s(\rho_s - \rho_{s-1})]$ and $\delta_s = \gamma_1 \cdots \gamma_s = 2^{-s}\exp(2^s \rho_s - \sum_{k=1}^{s-1} 2^k \rho_k - 2\rho_0)$. Let us show that

$$2^{-s} \log \delta_s \to 0 \quad \text{as} \quad s \to \infty. \tag{10}$$

Since $\rho_s \to 0$, we need to prove that $\sum_{k=1}^{s-1} 2^{k-s} \rho_k \to 0$ as $s \to \infty$. We can assume without loss of generality that the number s is odd, so $s = 2m+1$. Then, by monotonicity of (ρ_s), for the sum above we easily have $\sum_{k=1}^{2m} = \sum_{k=1}^{m} + \sum_{k=m+1}^{2m} \leq \rho_1 2^{-m} + \rho_{m+1}$, which converges to 0 as $m \to \infty$.

Given $s \in \mathbb{N}$, we uniformly distribute the mass 2^{-s} on each $I_{j,s}$ for $1 \leq j \leq 2^s$. We will denote by λ_s the normalized in this sense Lebesgue measure on E_s, so $d\lambda_s = (2^s l_{j,s})^{-1} dt$ on $I_{j,s}$.

If μ is a finite Borel measure of compact support then its logarithmic potential is defined by $U^{\mu}(z) = \int \log \frac{1}{|z-t|} d\mu(t)$. Let μ_K denote the equilibrium measure on a non-polar set K and $\xrightarrow{*}$ means convergence in the weak* topology.

Let $I = [a, b]$ with $b - a \leq 1$, $z \in I$. By partial integration,

$$\int_I \log \frac{1}{|z-t|} dt = b - a - (z-a)\log(z-a) - (b-z)\log(b-z).$$

It follows that

$$(b-a)\log\frac{e}{b-a} < \int_I \log\frac{1}{|z-t|} dt < (b-a)\log\frac{2e}{b-a}. \tag{11}$$

Lemma 7 *Let γ satisfy Eq. 8 and $R < \infty$. Then $U^{\lambda_s}(z) \to R$ for $z \in K(\gamma)$ as $s \to \infty$.*

Proof Fix $z \in K(\gamma)$. Given s, let $z \in I_{j,s}$ for $1 \le j \le 2^s$. From Eq. 11 we have $\int_{I_{j,s}} \log|z-t|^{-1}\, d\lambda_s(t) < 2^{-s}\,(2+\log l_{j,s}^{-1})$, which is $o(1)$ as $s \to \infty$, by Lemma 6 and Eq. 10.

To estimate $\sum_{k=1,k\ne j}^{2^s} \int_{I_{k,s}} \log|z-t|^{-1}\, d\lambda_s(t)$ we use $P_{2^s}(x) = \prod_{k=1}^{2^s}(x-y_k)$ with $y_k \in I_{k,s}$. As above, take the chain of basic intervals $I_{j,s} \subset I_{m,s-1} \subset I_{q,s-2} \subset \cdots \subset I_{1,0}$ containing z. Suppose k corresponds to the adjacent to $I_{j,s}$ subinterval $I_{k,s}$ of $I_{m,s-1}$. Then $h_{m,s-1} \le |z-t| \le |y_j - y_k| \le |z-t| + l_{j,s} + l_{k,s}$. Hence, $1 \le \frac{|y_j-y_k|}{|z-t|} \le 1+\varepsilon_0$, with $\varepsilon_0 = \frac{l_{j,s}+l_{k,s}}{h_{m,s-1}} < \frac{1}{7}$, by Lemma 4. For this k we get

$$2^{-s}\,\log|y_j - y_k|^{-1} < \int_{I_{k,s}} \log|z-t|^{-1}\, d\lambda_s(t) < 2^{-s}\,(\log|y_j-y_k|^{-1} + \varepsilon_0).$$

In its turn, $I_{q,s-2} \supset I_{m,s-1} \cup I_{n,s-1}$, where $I_{n,s-1}$ contains other two intervals of the $s-$th level. Let k correspond to any of them. Then $|z-t| - l_{j,s} - l_{k,s} \le |y_j - y_k| \le |z-t| + l_{j,s} + l_{k,s}$ with $|z-t| \ge h_{q,s-2}$. Here, $1-\varepsilon_1 \le \frac{|y_j-y_k|}{|z-t|} \le 1+\varepsilon_1$ with $\varepsilon_1 = \frac{l_{j,s}+l_{k,s}}{h_{q,s-2}} < \frac{8}{7}(\frac{l_{j,s}}{l_{m,s-1}}\,\frac{l_{m,s-1}}{l_{q,s-2}} + \frac{l_{k,s}}{l_{n,s-1}}\,\frac{l_{n,s-1}}{l_{q,s-2}}) < \frac{8}{7}\cdot 2\cdot 4\gamma_s\, 4\gamma_{s-1} < \frac{1}{7}\cdot\frac{1}{4}$, by Corollary 2. Repeating this argument leads to the representation

$$\sum_{k=1,k\ne j}^{2^s} \int_{I_{k,s}} \log|z-t|^{-1}\, d\lambda_s(t) = 2^{-s}\ \log \prod_{k=1,k\ne j}^{2^s} |y_j-y_k|^{-1} + \varepsilon,$$

where $|\varepsilon| \le 2^{-s+1}(\varepsilon_0 + 2\,\varepsilon_1 + \cdots + 2^{s-1}\,\varepsilon_{s-1})$ with $\varepsilon_k < \frac{2}{7}\cdot 8^{-k}$ for $k \ge 1$. Here we used the estimate $|\log(1+x)| \le 2\,|x|$ for $|x| < 1/2$. We see that $|\varepsilon| < 2^{-s}$.

The main term above is $2^{-s}\,\log|P'_{2^s}(y_j)|^{-1}$, which is $2^{-s}\,\log(\delta_s/r_s) + o(1)$, by Lemma 6. Thus,

$$\int \log|z-t|^{-1}\, d\lambda_s(t) = 2^{-s}\,\log(\delta_s/r_s) + o(1) \quad \text{as} \quad s \to \infty.$$

Finally, $2^{-s}\,\log(\delta_s/r_s) = R_s + 2^{-s}\,\log\frac{\delta_s}{2} \to R$ as $s \to \infty$, by Eq. 10. □

Theorem 4 *Suppose* γ *satisfies Eq.* 8 *and* $Cap(K(\gamma)) > 0$. *Then* $\lambda_s \overset{*}{\to} \mu_{K(\gamma)}$.

Proof All measures λ_s have unit mass. By Helly's Selection Theorem (see for instance [17], Theorem 0.1.3), we can select a subsequence $(\lambda_{s_k})_{k=1}^{\infty}$, weak* convergent to some measure μ. Approximating the function $\log|z-\cdot|^{-1}$ by the truncated continuous kernels (see for instance [17], Theorem 1.6.9), we get $\liminf_{k\to\infty} U^{\lambda_{s_k}}(z) = U^{\mu}(z)$ for quasi-every $z \in \mathbb{C}$. In particular, by Lemma 7, we have $U^{\mu}(z) = R$ for quasi-every $z \in K(\gamma)$. This means that $\mu = \mu_{K(\gamma)}$ (see e.g. [17], Theorem 1.3.3). The same proof remains valid for any subsequence $(\lambda_{s_j})_{j=1}^{\infty}$. Therefore, $\lambda_s \overset{*}{\to} \mu_{K(\gamma)}$. □

Suppose a non-polar Cantor-type set $K = \cap_{s=0}^{\infty} E_s$ with $E_s = \cup_{j=1}^{2^s} I_{j,s}$ is given and the measure λ_s is defined as above. Let us say that K is *weakly equilibrium* if $\lambda_s \overset{*}{\to} \mu_K$. On the other hand, let Λ_s be the normalized in the usual sense Lebesgue measure λ on E_s, so $d\Lambda_s = (\lambda E_s)^{-1} dt$ on E_s. We say that K is *equilibrium* if $\Lambda_s \overset{*}{\to} \mu_K$. The last means that the Cantor–Lebesgue measure λ_K coincides with the equilibrium measure μ_K. Of course, in the case of geometrically symmetric Cantor-type sets, when the lengths of all intervals of the $s-$th level are the same, there is no difference between λ_s and Λ_s and between the introduced features. Clearly, any compact set K with nonempty interior cannot be equilibrium in any sense since $supp\, \mu_K \subset \partial K$. Neither geometrically symmetric Cantor-type sets of positive capacity are equilibrium. For example, let us consider the set $K^{(\alpha)}$ from [1] which is constructed by means of the Cantor procedure with $l_{s+1} = l_s^\alpha$ for $1 < \alpha < 2$. The values $\alpha \geq 2$ give polar sets $K^{(\alpha)}$. Given $s \in \mathbb{N}$, let $z_s = l_1 - l_2 + \cdots + (-1)^{s+1} l_s$. Estimating distances $|z - t|$ for $z = 0$ and $z = z_s$, as in Lemma 7, it can be checked that $U^{\lambda_s}(0) - U^{\lambda_s}(z_s) > \sum_{k=1}^{s-1} 2^{-k-1} \log \frac{(l_{k-1}-l_k)(l_{k-1}-l_{k+1})}{(l_{k-1}-2l_k)(l_{k-1}-l_k-l_{k+1})}$. It is easily seen that all fractions in arguments of log exceed 1. Therefore, for each s there exists a point $z_s \in K^{(\alpha)}$ such that $U^{\lambda_s}(0) - U^{\lambda_s}(z_s)$ exceeds the constant $\frac{1}{4} \log \frac{(1-l_1)(1-l_2)}{(1-2l_1)(1-l_1-l_2)}$ and the limit logarithmic potential is not equilibrium. Indeed, if $K^{(\alpha)}$ is not polar, then it is regular with respect to the Dirichlet problem (see [13]) and $U^{\mu_{K^{(\alpha)}}}$ must be continuous in $\mathbb{C}$ and constant on $K^{(\alpha)}$.

Here we give the calculation without details, since a much stronger fact is valid for such sets and, in general, for certain Cantor repellers, where the equilibrium measure is supported by a set whose Hausdorff dimension is strictly smaller than the dimension of the whole set (see [4, 10, 12, 21]). Thus, the measures λ_K and μ_K are mutually singular in such cases.

Concerning our case, the question about convergence $\Lambda_s \overset{*}{\to} \mu_{K(\gamma)}$ is open. At least for some irregular cases, when $\gamma_k = \gamma_1$ for all k except $\gamma_{k_j} = \varepsilon_j$ with $\sum_{j=1}^{\infty} 2^{-k_j} \log \frac{1}{\varepsilon_j} < \infty$, the measures $\lambda_{K(\gamma)}$ and $\mu_{K(\gamma)}$ are different, so $K(\gamma)$ is not equilibrium.

Problem Construct, if it is possible, an equilibrium Cantor-type set.

7 Smoothness of $g_{\mathbb{C}\setminus K(\gamma)}$

We proceed to evaluate the modulus of continuity of the Green function corresponding to the set $K(\gamma)$. Recall that a modulus of continuity is a continuous nondecreasing subadditive function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$. Given function f, its modulus of continuity is $\omega(f, \delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|$.

In what follows the symbol $\sim$ denotes the strong equivalence: $a_s \sim b_s$ means that $a_s = b_s(1 + o(1))$ for $s \to \infty$. This gives a natural interpretation of the relation $\lesssim$.

Let γ be as in the preceding theorem. Then, we are given two monotone sequences $(\delta_s)_{s=1}^{\infty}$ and $(\rho_s)_{s=1}^{\infty}$ where, as above, $\delta_s = \gamma_1 \cdots \gamma_s$, $\rho_s = \sum_{k=s+1}^{\infty} 2^{-k} \log \frac{1}{2\gamma_k}$. We define the function ω by the following conditions: $\omega(0) = 0$, $\omega(\delta) = \rho_1$ for $\delta \geq \delta_1$. If $s \geq 2$ then $\omega(\delta) = \rho_s + 2^{-s} \log \frac{\delta}{\delta_s}$ for $\delta_s \leq \delta \leq \delta_{s-1}/16$ and $\omega(\delta) = \rho_{s-1} - k_s(\delta_{s-1} - \delta)$ for $\delta_{s-1}/16 < \delta < \delta_{s-1}$ with $k_s = \frac{16}{15} \cdot 2^{-s} \delta_{s-1}^{-1} \log 8$.

Lemma 8 *The function ω is a concave modulus of continuity. If $\gamma_s \to 0$ then for any positive constant C we have $\omega(\delta) \sim \rho_s + 2^{-s} \log \frac{C\delta}{\delta_s}$ as $\delta \to 0$ with $\delta_s \leq \delta < \delta_{s-1}$.*

Proof The function ω is continuous due to the choice of k_s. In addition, $\omega'(\delta_{s-1} + 0) < k_s < \omega'(\delta_{s-1}/16 - 0)$, which provides concavity of ω.

If $\gamma_s = \frac{1}{2}\exp[2^s(\rho_s - \rho_{s-1})] \to 0$ then $2^s \rho_s \to \infty$ and we have the desired equivalence in the case $\delta_s \le \delta \le \delta_{s-1}/16$. Suppose $\delta_{s-1}/16 < \delta < \delta_{s-1}$. The identity

$$\rho_{s-1} = \rho_s + 2^{-s}\,\log\frac{\delta_{s-1}}{2\delta_s} \tag{12}$$

yields $|\rho_s + 2^{-s}\,\log\frac{C\delta}{\delta_s} - \omega(\delta)| < 2^{-s}\big[|\log\frac{2C\delta}{\delta_{s-1}}| + \frac{16}{15}\log 8 \cdot \big(1 - \frac{\delta}{\delta_{s-1}}\big)\big] < 2^{-s}[|\log C| + 8\log 2]$, which is $o(\omega)$ since here $\omega(\delta) > \rho_{s-1} - 2^{-s}\log 8$. □

Lemma 9 *Suppose* γ *satisfies Eq.* 8 *and* $Cap(K(\gamma)) > 0$. *Let* $z \in \mathbb{C},\ z_0 \in K(\gamma)$ *with* $dist(z, K(\gamma)) = |z - z_0| = \delta < 1$. *Choose* $s \in \mathbb{N}$ *such that* $z_0 \in I_{j,s} \subset I_{j_1,s-1}$ *with* $l_{j,s} \le \delta < l_{j_1,s-1}$. *Then* $g_{\mathbb{C}\setminus K(\gamma)}(z) < \rho_s + 2^{-s}\,\log\frac{16\,\delta}{\delta_s}$.

On the other hand, if $l_{1,s} \le \delta < l_{1,s-1}$ *then* $g_{\mathbb{C}\setminus K(\gamma)}(-\delta) > \rho_s + 2^{-s}\,\log\frac{\delta}{\delta_s}$.

Proof Consider the chain of basic intervals containing z_0: $z_0 \in I_{j,s} \subset I_{j_1,s-1} \subset I_{j_2,s-2} \subset \cdots \subset I_{j_s,0} = [0, 1]$. Here, $I_{j_i,s-i} \setminus I_{j_{i-1},s-i+1}$ contains 2^{i-1} basic intervals of the $s-$th level. Each of them has certain endpoints x, y with $x \in X_s,\ y \in Y_{s-1}$. Recall that Y_{s-1} is the set of zeros of P_{2^s}. Distinguish $y_j \in I_{j,s}$. Now for a fixed large n we will express the value $|P_{2^n}(z)| = \prod_{k=1}^{2^n}|z - x_k|$ in terms of $\prod_{k=1,k\neq j}^{2^s}|y_j - y_k|$ (compare to Lemma 7). Clearly, each interval of the $s-$th level contains 2^{n-s} zeros of P_{2^n}, so we will replace these 2^{n-s} points with the corresponding y_k.

Let us first consider the product $\pi_0 := \prod_{x_k \in I_{j,s}}|z - x_k|$. Here, $|z - x_k| \le \delta + l_{j,s} < 2\,\delta$, so $\pi_0 < (2\,\delta)^{2^{n-s}}$.

Let $\pi_1 := \prod_{x_k \in I_{m,s}}|z - x_k|$, where $I_{m,s}$ is adjacent to $I_{j,s}$. Then $|z_0 - x_k| \le l_{j_1,s-1} = |y_j - y_m|$, since y_j and y_m are the endpoints of the interval $I_{j_1,s-1}$. Therefore, $|z - x_k| < 2\,|y_j - y_m|$ and $\pi_1 < (2\,|y_j - y_m|)^{2^{n-s}}$.

In the general case, given $2 \le i \le s$, let π_i denote the product of all $|z - x_k|$ for $x_k \in J_i := I_{j_i,s-i} \setminus I_{j_{i-1},s-i+1}$. Suppose $x_k \in I_{q,s}$. Then, $|z - x_k| \le \delta + l_{j,s} + |y_j - y_q| + l_{q,s} \le |y_j - y_q|(1 + \frac{\delta + l_{j,s} + l_{q,s}}{h_{j_i,s-i}})$, since y_j and y_q belong to different subintervals of the $(s - i + 1)-$th level for $I_{j_i,s-i}$. Here, $\frac{\delta}{h_{j_i,s-i}} < \frac{8}{7}\,\frac{l_{j_1,s-1}}{l_{j_i,s-i}} < \frac{8}{7}\,8^{1-i}$, by Corollary 2. As in the proof of Lemma 7, we obtain $\frac{l_{j,s} + l_{q,s}}{h_{j_i,s-i}} < \frac{8}{7}\cdot 2 \cdot 8^{-i}$. From this, $\prod_{x_k \in I_{q,s}}|z - x_k| \le [\,|y_j - y_q|\,(1 + \frac{80}{7}\,8^{-i})]^{2^{n-s}}$. Since J_i contains 2^{i-1} basic intervals of the $s-$th level, $\pi_i < [\,(1 + \frac{80}{7}\,8^{-i})^{2^{i-1}}\prod_{y_q \in J_i}|y_j - y_q|\,]^{2^{n-s}}$.

The product $\prod_{i=2}^{s}(1 + \frac{80}{7}\,8^{-i})^{2^{i-1}}$ is smaller than 2, as is easy to check.

Therefore, $|P_{2^n}(z)| = \prod_{i=0}^{s}\pi_i < [\,8 \cdot \delta \cdot \prod_{k=1,k\neq j}^{2^s}|y_j - y_k|\,]^{2^{n-s}}$. The last product in the square brackets is $|P'_{2^s}(y_j)|$, which does not exceed r_s/δ_s, by Lemma 6. Hence, $2^{-n}\,\log|P_{2^n}(z)| < 2^{-s}\,\log\frac{16\,\delta}{\delta_s} - R_s$.

Finally, by Corollary 1, $g_{\mathbb{C}\setminus K(\gamma)}(z) = R + \lim_{n\to\infty} 2^{-n}\,\log|P_{2^n}(z)|$, which yields the desired upper bound of the Green function.

Similar, but simpler calculations establish the sharpness of the bound. We have $g_{\mathbb{C}\setminus K(\gamma)}(-\delta) = R + \lim_{n\to\infty} 2^{-n}\log P_{2^n}(-\delta)$. Now, $P_{2^n}(-\delta) = \prod_{i=0}^{s}\pi_i$ with $\pi_0 = \prod_{x_k \in I_{1,s}}(\delta + x_k) > \delta^{2^{n-s}}$ and $\pi_i = \prod_{x_k \in I_{2,s-i+1}}(\delta + x_k)$ for $i \ge 1$. Suppose $x_k \in I_{q,s} \subset I_{2,s-i+1}$. Then $\delta + x_k > y_q - l_{q,s}$. Since $y_q > h_{1,s-i} > \frac{7}{8}l_{1,s-i}$, we have $\delta + x_k > y_q(1 - \frac{8}{7}\,8^{-i})$ and $\pi_i > [\,(1 - \frac{1}{7}\,8^{1-i})^{2^{i-1}}\prod_{y_q \in I_{2,s-i+1}} y_q\,]^{2^{n-s}}$. Therefore,

$P_{2^n}(-\delta) > [\,\frac{\delta}{2}\prod_{k=1}^{2^s} y_k]^{2^{n-s}} = [\,\frac{\delta}{2}\,|P'_{2^s}(0)|]^{2^{n-s}} = [\,\delta/\delta_s \cdot r_s/2\,]^{2^{n-s}}$, by Eq. 6. Thus, $2^{-n}\log P_{2^n}(-\delta) > -R_s + 2^{-s}\log\frac{\delta}{\delta_s}$ and $g_{\mathbb{C}\setminus K(\gamma)}(-\delta) \geq \rho_s + 2^{-s}\log\frac{\delta}{\delta_s}$. □

Theorem 5 *Suppose* γ *satisfies Eq.* 8 *and* $Cap(K(\gamma)) > 0$. *If* $\delta_s \leq \delta < \delta_{s-1}$ *then* $\rho_s + 2^{-s}\log\frac{\delta}{\delta_s} < \omega(g_{\mathbb{C}\setminus K(\gamma)}, \delta) < \rho_s + 2^{-s}\log\frac{16\,\delta}{\delta_s}$. *If* $\gamma_s \to 0$ *then* $\omega(g_{\mathbb{C}\setminus K(\gamma)}, \delta) \sim \omega(\delta)$ *as* $\delta \to 0$.

Proof Fix δ and s with $\delta_s \leq \delta < \delta_{s-1}$. By Eq. 9, $\delta_s < l_{1,s} < 2\,\delta_s < \delta_{s-1}$.

If $l_{1,s} \leq \delta < \delta_{s-1}$ then $\omega(g_{\mathbb{C}\setminus K(\gamma)}, \delta) \geq g_{\mathbb{C}\setminus K(\gamma)}(-\delta)$, so Lemma 9 yields the desired lower bound. If $\delta_s \leq \delta < l_{1,s}$, then $g_{\mathbb{C}\setminus K(\gamma)}(-\delta) > \rho_{s+1} + 2^{-s-1}\log\frac{\delta}{\delta_{s+1}} = \rho_s + 2^{-s-1}\log\frac{2\,\delta}{\delta_s}$, by Eq. 12. Here, $2^{-s-1}\log\frac{2\,\delta}{\delta_s} > 2^{-s}\log\frac{2\,\delta}{\delta_s}$, as is easy to check.

In order to get the upper bound, it is enough to estimate $g_{\mathbb{C}\setminus K(\gamma)}(z)$ for $z \in \mathbb{C}$ with $dist(z, K(\gamma)) = \delta$. Indeed, the modulus of continuity of $g_{\mathbb{C}\setminus K}$ is realized on the boundary of K (see e.g. 3.6 in [18]).

Let us fix $z_0 \in K(\gamma)$ such that $dist(z, K(\gamma)) = |z - z_0|$.

Fix m such that $z_0 \in I_{j,m} \subset I_{j_1,m-1}$ for some j with $l_{j,m} \leq \delta < l_{j_1,m-1}$. Then $m \geq s$, since otherwise Lemma 6 gives a contradiction $\delta < \delta_{s-1} \leq \delta_m < l_{j,m} \leq \delta$.

If $m = s$ then, by Lemma 9, the result is immediate.

If $m \geq s+1$ then $g_{\mathbb{C}\setminus K(\gamma)}(z) \leq \rho_m + 2^{-m}\log\frac{16\,\delta}{\delta_m}$ that does not exceed $\rho_s + 2^{-s}\log\frac{16\,\delta}{\delta_s}$. Indeed, the function $f(\delta) = \rho_s - \rho_m + (2^{-s} - 2^{-m})\log 16\,\delta - 2^{-s}\log\delta_s + 2^{-m}\log\delta_m$ attains its minimal value on $[\delta_s, \delta_{s-1})$ at the left endpoint. Here, $f(\delta_s) = (2^{-s} - 2^{-m})\log 8 + \sum_{k=s+1}^{m}(2^{-k} - 2^{-m})\log\frac{1}{\gamma_k} > 0$.

The last statement of the theorem is a corollary of Lemma 8. □

8 Model Types of Smoothness

Let us consider some model examples with different rates of decrease of $(\rho_s)_{s=1}^{\infty}$. Recall that for non-polar sets $K(\gamma)$ we have $R = Rob(K(\gamma)) = \sum_{k=1}^{\infty} 2^{-k}\log\frac{1}{\gamma_k}$. Here, $\rho_s = \sum_{k=s+1}^{\infty} 2^{-k}\log\frac{1}{2\gamma_k}$ shows how rapidly $Rob(\overline{D}_s)$ approximates R. From Eq. 8 it follows that $\rho_s \geq 2^{-s}\log 16$ and $R \geq \log 32$, so $Cap(K(\gamma)) \leq 1/32$.

If a set K is uniformly perfect, then the function $g_{\mathbb{C}\setminus K}$ is Hölder continuous (see e.g. [10, p. 119]), which means the existence of constants C, α such that

$$g_{\mathbb{C}\setminus K}(z) \leq C\,(dist(z, K))^{\alpha} \quad \text{for all} \quad z \in \mathbb{C}.$$

In this case we write $g_{\mathbb{C}\setminus K} \in Lip\ \alpha$.

By Theorem 3, $g_{\mathbb{C}\setminus K(\gamma)}$ is Hölder continuous provided $\gamma_s = const$. Now we can control the exponent α in the definition above. In the following examples we suppose that $dist(z, K(\gamma)) = \delta$ with $\delta_s \leq \delta < \delta_{s-1}$ for large s.

Example 2 Let $\gamma_s = \gamma_1 \leq \frac{1}{32}$ for all s. Then $\delta_s = \gamma_1^s, r_s = \gamma_1^{2^s-1}$, $R = \log\frac{1}{\gamma_1}$, and $\rho_s = 2^{-s}\log\frac{1}{2\gamma_1}$. Here, $\rho_s + 2^{-s}\log\frac{\delta}{\delta_s} \geq \rho_s > 2^{-s} = \delta_s^{\alpha}$ with $\alpha = -\frac{\log 2}{\log\gamma_1}$. Since $\delta_s = \gamma_1\,\delta_{s-1} > \gamma_1\,\delta$, we have, by Theorem 5, $g_{\mathbb{C}\setminus K(\gamma)}(-\delta) > \gamma_1^{\alpha}\,\delta^{\alpha}$. On the other hand, $\omega(g_{\mathbb{C}\setminus K(\gamma)}, \delta) < \rho_s + 2^{-s}\log\frac{16\delta}{\delta_s} < \delta^{\alpha}\log\frac{8}{\gamma_1^2}$.

Suppose we are given α with $0 < \alpha \leq 1/5$. Then the value $\gamma_s = 2^{-1/\alpha}$ for all s provides $g_{\mathbb{C}\setminus K(\gamma)} \in Lip\ \alpha$ and $g_{\mathbb{C}\setminus K(\gamma)} \notin Lip\ \beta$ for $\beta > \alpha$.

The next example is related to the function $h(\delta) = (\log \frac{1}{\delta})^{-1}$ that defines the logarithmic measure of sets. Let us write $g_{\mathbb{C}\setminus K} \in Lip_h\, \alpha$ if for some constants C we have

$$g_{\mathbb{C}\setminus K}(z) \le C\, h^{\alpha}(dist(z, K)) \quad \text{for all} \quad z \in \mathbb{C}.$$

Example 3 Given $1/2 < \rho < 1$, let $\rho_s = \rho^s$ for $s \ge s_0$, where $\frac{\rho}{1-\rho} \log 16 < (2\rho)^{s_0}$. This condition provides $\gamma_s < 1/32$ for $s > s_0$. Suppose $\gamma_s = 1/32$ for $s \le s_0$, so we can use Theorem 5. For large s we have $\delta_s = C\, 2^{-s} \mu^{(2\rho)^s}$ with $\mu = \exp(\frac{2\rho-2}{2\rho-1})$ and some constant C. Let us take $\alpha = \frac{\log(1/\rho)}{\log(2\rho)}$, so $(2\rho)^{\alpha} = 1/\rho$. Then $h^{\alpha}(\delta) \ge h^{\alpha}(\delta_s) \ge \varepsilon_0 (2\rho)^{-s\alpha} = \varepsilon_0\, \rho \cdot \rho_{s-1}$ for some ε_0. From this we conclude that $g_{\mathbb{C}\setminus K(\gamma)} \in Lip_h\, \alpha$ for given α. Evaluation $g_{\mathbb{C}\setminus K(\gamma)}(-\delta_s)$ from below yields $g_{\mathbb{C}\setminus K(\gamma)} \notin Lip_h\, \beta$ for $\beta > \alpha$. Now, given $\alpha > 0$, the value $\rho = 2^{-\frac{\alpha}{1+\alpha}}$ provides the Green function of the exact class $Lip_h\, \alpha$ (compare this to [1, 8]).

Example 4 Let $\rho_s = 1/s$. Then $\gamma_s = \frac{1}{2}\exp(\frac{-2^s}{s^2-s}) < 1/32$ for $s \ge 8$. As above, all previous values of γ_s are 1/32. Here, $\delta_s = C\, 2^{-s} \exp\left[\frac{2^s}{s} - \sum_{k=1}^{s-1} \frac{2^k}{k}\right]$. Summation by parts (see e.g. [16], Theorem 3.41) yields $\delta_s = C\, 2^{-s} \exp[-2^{s+1}(s^{-2} + o(s^{-2}))]$. From this, $\omega(g_{\mathbb{C}\setminus K(\gamma)}, \delta) \sim \frac{1}{s} \sim \frac{\log 2}{\log\log 1/\delta_s}$.

Example 5 Here we present Cantor-type sets $K(\gamma)$ with "lowest smoothness" of the corresponding Green function. Given $N \in \mathbb{N}$, let $F_N(t) = \log\log \cdots \log t$ be the $N-$th iteration of the logarithmic function. Let $\rho_s = (F_N(s))^{-1}$ for large enough s. Here, $\rho_{k-1} - \rho_k \sim [k \cdot \log k \cdot F_2(k) \cdots F_{N-1}(k) \cdot F_N^2(k)]^{-1}$. Since $\delta_s = C\, 2^{-s} \exp[-\sum_{k=1}^{s} 2^k (\rho_{k-1} - \rho_k)]$, we have, as above, $s \sim \frac{\log\log 1/\delta_s}{\log 2}$. Thus, $\omega(g_{\mathbb{C}\setminus K(\gamma)}, \delta) \sim [F_{N+2}(1/\delta)]^{-1}$.

We see that a slower decrease of (ρ_s) implies a less smooth $g_{\mathbb{C}\setminus K(\gamma)}$ and conversely. If, in examples above, we take $\gamma_s = 1/32$ for $s < s_0$ with rather large s_0, then the set $K(\gamma)$ will have logarithmic capacity as close to 1/32, as we wish.

Problem Given modulus of continuity ω, to find $(\gamma_s)_{s=1}^{\infty}$ such that $\omega(g_{\mathbb{C}\setminus K(\gamma)}, \cdot)$ coincides with ω at least on some null sequence.

9 Markov's Factors

For any infinite compact set $K \subset \mathbb{C}$ we consider the sequence of Markov's factors $M_n(K) = \inf\{M : \ |P'|_K \le M\, |P|_K \ \text{ for all } \ P \in \mathcal{P}_n\}$, $n \in \mathbb{N}$. We see that $M_n(K)$ is the norm of the operator of differentiation in the space $(\mathcal{P}_n, |\cdot|_K)$. In the case of non-polar K, the knowledge about smoothness of the Green function near the boundary of K may help to estimate $M_n(K)$ from above. The application of the Cauchy formula for P' and the Bernstein–Walsh inequality yields the estimate

$$M_n(K) \le \inf_{\delta} \delta^{-1} \exp[n \cdot \omega(g_{\mathbb{C}\setminus K}, \delta)]. \tag{13}$$

This approach gives an effective bound of $M_n(K)$ for the cases of temperate growth of $\omega(g_{\mathbb{C}\setminus K}, \cdot)$. For instance, the Hölder continuity of $g_{\mathbb{C}\setminus K}$ implies Markov's property of the set K, which means that there are constants C, m such that $M_n(K) \le Cn^m$ for all n.

Lemma 10 *Suppose* γ *satisfies Eq.* 8 *and* $Cap(K(\gamma)) > 0$. *Given fixed* $s \in \mathbb{N}$, *let* $f(\delta) = \delta^{-1}\exp[2^s(\rho_k + 2^{-k}\log\frac{16\,\delta}{\delta_k})]$ *for* $\delta_k \le \delta < \delta_{k-1}$ *with* $k \ge 2$. *Then* $\inf_{0<\delta<\delta_1} f(\delta) = f(\delta_s - 0) = 4\sqrt{2}\,\delta_s^{-1}\exp(2^s\,\rho_s)$.

Proof Let us fix the interval $I_k = [\delta_k, \delta_{k-1})$. In view of the representation $f(\delta) = C_{s,k}\,\delta^{2^{s-k}-1}$, the function f increases for $k < s$, decreases for $k > s$, and is constant for $k = s$ on I_k. An easy computation shows that $f(\delta_{k+1}) < f(\delta_k)$ for $k \le s-1$ and $f(\delta_{k-1} - 0) < f(\delta_k - 0)$ for $k \ge s+1$. Thus, it remains to compare $f(\delta_s - 0)$ and $f(\delta_s)$. Here, $f(\delta_s) = 16\,\delta_s^{-1}\exp(2^s\,\rho_s)$ exceeds $f(\delta_s - 0) = \delta_s^{-1}(16/\gamma_{s+1})^{1/2}\exp(2^s\rho_{s+1}) = 4\sqrt{2}\,\delta_s^{-1}\exp(2^s\,\rho_s)$. □

Example 6 Let $\gamma_s = \gamma_1 \le \frac{1}{32}$ for $s \in \mathbb{N}$. Lemma 10 and Example 2 imply $M_{2^s}(K(\gamma)) \le \sqrt{8}\cdot\delta_{s+1}^{-1} = \sqrt{8}\,\gamma_1^{-1}\,2^{s/\alpha}$, where α is the same as in Example 2.

On the other hand, let $Q = P_{2^s} + r_s/2$. Then $|Q|_{K(\gamma)} = r_s/2$ and $|Q'(0)| = r_s/\delta_s$, so $M_{2^s}(K(\gamma)) \ge 2\,\delta_s^{-1} = 2\cdot 2^{s/\alpha}$. Now, for each n we choose s with $2^s \le n < 2^{s+1}$. Since the sequence of Markov's factors increases,

$$c\,n^{1/\alpha} \le M_{2^s}(K(\gamma)) \le M_n(K(\gamma)) \le M_{2^{s+1}}(K(\gamma)) \le C\,n^{1/\alpha}$$

with $c = 2^{1-1/\alpha}$, $C = \gamma_1^{-1}\,2^{3/2+1/\alpha}$. Given $m \in [5, \infty)$, the value $\gamma_s = 2^{-m}$ for all s provides the set $K(\gamma)$ with the best Markov's exponent $m(K(\gamma)) = m = 1/\alpha$.

However, the estimate Eq. 13 may be rather rough for compact sets with less smooth moduli of continuity of the corresponding Green's functions. For instance, let us consider the set $K(\gamma)$ with $\sum_{k=1}^{\infty}\gamma_k < \infty$. Since $\gamma_k = \frac{1}{2}\exp[2^k(\rho_k - \rho_{k-1})]$, we have $2^k(\rho_{k-1} - \rho_k) \to \infty$ and $2^k\,\rho_k \to \infty$. By Lemma 10, the exact value of the right side in Eq. 13 for $n = 2^s$ is $4\sqrt{2}\,\delta_s^{-1}\exp(2^s\,\rho_s)$, whereas $M_{2^s}(K(\gamma)) \sim 2\,\delta_s^{-1}$, which will be shown below by means of the Lagrange interpolation. It should be noted that the set $K(\gamma)$ may be polar here.

Let us interpolate $P \in \mathcal{P}_{2^s}$ at zeros $(x_k)_{k=1}^{2^s}$ of P_{2^s} and at one extra point $l_{1,s}$. Then the fundamental Lagrange interpolating polynomials are $L_*(x) = -P_{2^s}(x)/r_s$ and $L_k(x) = \frac{(x-l_{1,s})P_{2^s}(x)}{(x-x_k)(x_k-l_{1,s})P'_{2^s}(x_k)}$ for $k = 1, 2, \cdots, 2^s$. Let Δ_s denote $\sup_{x\in K(\gamma)}[\,|L'_*(x)| + \sum_{k=1}^{2^s}|L'_k(x)|\,]$. For convenience we enumerate $(x_k)_{k=1}^{2^s}$ in increasing way, so $x_k \in I_{k,s}$ for $1 \le k \le 2^s$.

Lemma 11 *Suppose* γ *satisfies Eq.* 8 *and* $\sum_{k=1}^{\infty}\gamma_k < \infty$. *Then* $\Delta_s \sim 2\,\delta_s^{-1}$.

Proof We use the following representation:

$$L'_k(x) = \frac{P'_{2^s}(x)}{(x_k - l_{1,s})P'_{2^s}(x_k)} + \frac{P_{2^s}(x)}{(x - x_k)P'_{2^s}(x_k)}\sum_{j=1,\,j\ne k}^{2^s}\frac{1}{x - x_j} =: A_k + B_k. \tag{14}$$

In particular, $L'_1(0) = -l_{1,s}^{-1} - \sum_{j=2}^{2^s} x_k^{-1}$. By Eq. 6, $|L'_*(0)| = \delta_s^{-1}$, so $\Delta_s > |L'_*(0)| + |L'_1(0)| > \delta_s^{-1} + l_{1,s}^{-1} > \delta_s^{-1}(1 + e^{-16\gamma_s})$, by Eq. 9. Thus, $\Delta_s \gtrsim 2\,\delta_s^{-1}$.

We proceed to estimate Δ_s from above. Lemma 6 gives the uniform bound $|L'_*(x)| \le \delta_s^{-1}$.

Let us examine separately the sum $\sum_{k=1}^{2^s} |A_k|$, where A_k are defined by Eq. 14. Let $C_0 = \exp(16 \sum_{k=1}^{\infty} \gamma_k)$. Then, by Lemma 6, $|P'_{2^s}(x)| \le |P'_{2^s}(0)| = r_s/\delta_s < C_0 |P'_{2^s}(x_k)|$ for $x \in K(\gamma)$. Therefore, $|A_1| \le l_{1,s}^{-1} < \delta_s^{-1}$ and $\sum_{k=2}^{2^s} |A_k| < C_0 \sum_{k=2}^{2^s} (x_k - l_{1,s})^{-1}$. Here, $\sum_{k=2}^{2^s} (x_k - l_{1,s})^{-1} < 2\, l_{1,s-1}^{-1}$, as is easy to check. Thus, $\sum_{k=1}^{2^s} |A_k| < \delta_s^{-1} + 2C_0 \delta_{s-1}^{-1}$.

In order to estimate the sum of the addends B_k, let us fix $x \in K(\gamma)$ and $1 \le m \le 2^s$ such that $x \in I_{m,s}$. Suppose first that $k \neq m$. Then

$$\sum_{j=1,\, j\neq k}^{2^s} \left| \frac{P_{2^s}(x)}{x - x_j} \right| < 2 \left| \frac{P_{2^s}(x)}{x - x_m} \right| \le 2\, |P'_{2^s}(\xi)| \tag{15}$$

with a certain $\xi \in I_{m,s}$. Indeed, if $x = x_m$ then this sum is exactly $|P'_{2^s}(x_m)|$, so $\xi = x_m$. Otherwise we take the main term out of the brackets:

$$\left| \frac{P_{2^s}(x)}{x - x_m} \right| \left[1 + \sum_{j=1,\, j\neq k,\, j\neq m}^{2^s} \left| \frac{x - x_m}{x - x_j} \right| \right].$$

Here the sum in the square brackets can be handled in the same way as in the proof of Lemma 4. Let $I_{m,s} \subset I_{q,s-1} \subset I_{r,s-2} \subset \cdots$. Then $[\cdots] \le 1 + l_{m,s}(h_{q,s-1}^{-1} + 2h_{r,s-2}^{-1} + \cdots) \le 1 + \frac{8}{7} l_{m,s}(l_{q,s-1}^{-1} + 2l_{r,s-2}^{-1} + \cdots) < 1 + \frac{8}{7}(4\gamma_s + 2 \cdot 4\gamma_s 4\gamma_{s-1} + \cdots) < 2$.

On the other hand, by Taylor's formula, $P_{2^s}(x) = P'_{2^s}(\xi)(x - x_m)$ with $\xi \in I_{m,s}$, which establishes Eq. 15.

Therefore,

$$\sum_{k=1,k\neq m}^{2^s} |B_k| < \sum_{k=1,k\neq m}^{2^s} \frac{2\,C_0}{|x - x_k|}.$$

As above, $\sum_{k=1,k\neq m}^{2^s} |B_k| < 2\,C_0(h_{q,s-1}^{-1} + 2h_{r,s-2}^{-1} + \cdots) < 4\,C_0\, h_{q,s-1}^{-1} < 5\,C_0\, l_{q,s-1}^{-1}$.

It remains to consider $B_m = \frac{P_{2^s}(x)}{(x - x_m) P'_{2^s}(x_m)} \sum_{j=1,\, j\neq m}^{2^s} \frac{1}{x - x_j}$. Let us take the interval $I_{n,s}$ adjacent to $I_{m,s}$, so $I_{n,s} \cup I_{m,s} \subset I_{q,s-1}$. Then, as above, $\sum_{j=1,\, j\neq m}^{2^s} |x - x_j|^{-1} < 2\,|x - x_n|^{-1}$ and $|B_m| < 2\,C_0\,|x - x_n|^{-1} < 3\,C_0\, l_{q,s-1}^{-1}$, since $|x - x_n| > h_{q,s-1}$.

This gives $\sum_{k=1}^{2^s} |B_k| < 8\,C_0\, l_{q,s-1}^{-1} < 8\,C_0\, \delta_{s-1}^{-1}$, by Lemma 6. Finally, $\Delta_s < 2\,\delta_s^{-1} + 10\,C_0\, \delta_{s-1}^{-1} = \delta_s^{-1}(2 + 10\,C_0\,\gamma_s) \sim 2\,\delta_s^{-1}$. □

Theorem 6 *With the assumptions of Lemma* 11, $M_{2^s}(K(\gamma)) \sim 2\,\delta_s^{-1}$.

Proof On the one hand, $|P_{2^s} + r_s/2|_{K(\gamma)} = r_s/2$ and $|P'_{2^s}(0)| = r_s/\delta_s$, so $M_{2^s}(K(\gamma)) \ge 2\,\delta_s^{-1}$.

On the other hand, for each polynomial $P \in \mathcal{P}_{2^s}$ and $x \in K(\gamma)$ we have $|P'(x)| \le |P|_{K(\gamma)}\, \Delta_s$, and the theorem follows. □

We are now in a position to construct a compact set with preassigned growth of subsequence of Markov's factors. Suppose we are given a sequence of positive terms $(M_{2^s})_{s=0}^{\infty}$ with $\sum_{s=0}^{\infty} M_{2^s}/M_{2^{s+1}} < \infty$. The case of polynomial growth of (M_n)

was considered before, so let us assume that $C n^m M_n^{-1} \to 0$ as $n \to \infty$ for fixed C and m. Fix s_0 such that $M_{2^s}/M_{2^{s+1}} \le 1/32$ for $s \ge s_0$ and $M_{2^{s_0}} \ge 2 \cdot 2^{5s_0}$.

Let us take $\gamma_s = M_{2^{s-1}}/M_{2^s}$ for $s > s_0$ and $\gamma_s = (2/M_{2^{s_0}})^{1/s_0}$ for $s \le s_0$. Then $\gamma_s \le 1/32$ for all s and we can use Theorem 6. Here, $\delta_s = 2/M_{2^s}$, so $M_{2^s}(K(\gamma)) \sim M_{2^s}$.

It should be noted that the growth of $(M_n(K))$ is restricted for a non-polar compact set K ([6], Proposition 3.1). It is also interesting to compare Theorem 6 with Theorem 2 in [19].

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